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1976 J. Phys. A: Math. Gen. 9 1145

(http://iopscience.iop.org/0305-4470/9/7/017)

J. Phys. A: Math. Gen., Vol. 9, No. 7, 1976. Printed in Great Britain. © 1976

Energy expressions and energy velocity for wave packets in an absorptive and dispersive medium

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Received 20 October 1975, in final form 3 February 1976

Abstract. Expressions for the stored energy, energy power, power dissipated and the velocity of energy propagation are derived for wave pulses with slowly varying amplitudes. The results are expressed in terms of the dispersion function $D(\omega, k, \lambda)$ which is assumed to be an explicit function of the dissipational parameter λ. It is shown that in the case of resonant pulse propagation through a non-inverted atomic medium the energy velocity is always less than the free space velocity of light c. It is also shown, using the expression for the dissipated power in an atomic medium, that the asymmetric absorption of energy of the pulse results in a motion of the pulse maximum at a velocity greater than c.

1. Introduction

Expressions for the stored energy, energy flow and the velocity of energy propagation are of basic interest in the analysis of wave motion.

For propagation of wave pulses through a non-absorbing but strongly dispersive medium these quantities can be obtained from a Lagrangian density (cf Anderson and Askne 1972, 1974) including correction terms associated with the dispersive properties of the medium as well as with the variation of the slowly varying amplitude.

For an absorbing medium, on the other hand, difficulties arise in attempting to separate the energy components and to derive the energy expressions as well as the energy velocity. In the case of propagation of quasi-monochromatic waves in a specific medium, energy expressions are easily calculated starting from Maxwell's equations (see, for example, Brillouin 1960, Loudon 1970). However, for wave pulses with slowly varying amplitudes it is difficult to determine the evolution of the pulse envelope and to solve Maxwell's equations. Frequently used expressions for the stored energy and the energy flow are based on an assumption of small losses (cf Ginzburg 1964). The derivation of the energy expressions corresponding to propagation of quasi-monochromatic waves in the presence of absorption and temporal dispersion is presented in Askne and Lind (1970), where the analysis is not restricted to media with small losses.

The aim of the present paper is to derive the generalized energy expressions for waves with slowly varying amplitudes in an absorptive and strongly dispersive medium, which include higher-order correction terms that are essential when the medium is absorptive and strongly dispersive and the variation of the slowly varying amplitude no longer can be neglected. The resulting stored energy, energy flow, and power dissipated are expressed in terms of the dispersion function D, which is an explicit function of the dissipational parameter λ. The analysis is restricted to an isotropic, homogeneous, linear and reciprocal medium. However, the medium can be temporally as well as spatially dispersive. Specifying the results for a Gaussian pulse, we obtain the velocity of energy propagation v_E defined as the rate of change of the 'temporal centre' of energy flow. It will also be shown that the second-order generalized energy velocity includes the second-order expressions for the 'temporal' pulse velocity of propagation of a Gaussian pulse in an absorbing medium (cf Anderson *et al* 1975), and the velocity of the moment of inertia for the case of a non-absorbing medium (cf Anderson and Askne 1974). Application to a non-inverted atomic medium shows that the energy velocity is always less than the free-space velocity of light c. Finally, we will use the expression for the power dissipated to show that the asymmetric absorption of energy results in a forward motion of the centre of energy of the pulse so that the maximum of the pulse envelope can move with speed greater than c.

2. Energy expressions for slowly varying waves in an absorbing medium

We will show that it is possible to derive generalized energy expressions from the dispersion function if this is an explicit function of the dissipational parameter λ. In order to avoid unneccesary complications we will specialize to an isotropic medium. We assume further a linear, homogeneous, dispersive and one-dimensional wave problem characterized by the following matrix equation:

$$\mathbf{A}\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \lambda\right)\boldsymbol{E}(t, x) = \boldsymbol{J}^{s}(t, x). \tag{1}$$

x and t denote space and time coordinates, $\mathbf{A}$ is a linear matrix operator, and $\boldsymbol{E}$ and $\boldsymbol{J}^{s}$ are vectors specifying the wave fields and the external driving fields respectively. If we have one external field only, a 'current' density $\boldsymbol{J}^{s}$ (t, x), we can Fourier transform equation (1) into the scalar form (cf Askne 1972)

$$D(\omega, k, \lambda)E(\omega, k) = \mathrm{j}J^{s}(\omega, k), \tag{2}$$

where $D(\omega, k, \lambda)$ is the dispersion function, $E(\omega, k)$ and $J^{s}(\omega, k)$ are the transforms of a fundamental wave field $E(t, x)$ and the current density $J^{s}(t, x)$. $E(t, x)$ is chosen such that the product $-E(t, x)J^{s}(t, x)$ constitutes the power (per volume element) delivered by the external source. For a full account of these matters see Askne (1972). Notations and terminology are chosen with a possible application to the electromagnetic waves in mind. However, this is not essential to the analysis, e.g. in a mechanical wave problem the corresponding quantities would have been the velocity field and the force density.

In order to obtain energy expressions we restrict ourselves to media with moderate absorption, i.e. the real part of the wavenumber k_{re} is one order of magnitude larger than the imaginary part of the wavenumber k_{im}, which is true in most cases of practical interest. Assuming the waves with slowly varying amplitudes

$$\begin{aligned} E(t, x) &= E_0(t, x) \exp[\mathrm{j}(\omega_0 t - k_{0re}x)], \\ J^{s}(t, x) &= J^{s}_0(t, x) \exp[\mathrm{j}(\omega_0 t - k_{0re}x)], \end{aligned} \tag{3}$$

where $E_0(t, x)$ and $J^{s}_0(t, x)$ are slowly varying amplitudes compared to $\exp[\mathrm{j}(\omega_0 t -$

$k_{0re}x)]$, the relation (2) can be written as (cf Askne 1972)

$$D\left(\omega_0 - \mathrm{j}\frac{\partial}{\partial t},\ k_{0re} + \mathrm{j}\frac{\partial}{\partial x},\ \lambda\right)E_0(t, x) = \mathrm{j}J_0^s(t, x), \tag{4}$$

where $D(\omega_0 - \mathrm{j}(\partial/\partial t),\ k_{0re} + \mathrm{j}(\partial/\partial x),\ \lambda)$ can be interpreted as a dispersion operator given by a Taylor expansion around ω_0 and k_{0re}.

The energy components are assumed to satisfy the energy conservation law

$$\frac{\partial W}{\partial t} + \frac{\partial S}{\partial x} + P_d = P_s, \tag{5}$$

where W is the averaged stored energy, S the averaged energy flow, P_d the averaged power dissipated, and P_s the averaged power delivered by the external source. The main problem is to separate the energy components. We will now consider some cases when it is possible to derive expressions for P_d and P_s which together with (5) yield expressions for W and S.

2.1. The power dissipated

In order to separate the energy components we need an expression for the power dissipated which can be derived from the dispersion function D if this is a known rational function of the loss parameter λ (cf Askne and Lind 1970). In a model description of a medium the dissipation is often characterized by a damping 'frictional' force associated with an oscillatory velocity. The latter is denoted by $v_0(t, x)\exp[\mathrm{j}(\omega_0 t - k_{0re}x)]$. The frictional force may be assumed to be proportional to the velocity, i.e. $\lambda v_0(t, x)\exp[\mathrm{j}(\omega_0 t - k_{0re}x)]$, where λ is frequency and wavelength independent. If the dissipation is obtained in a part of the system with no relative motion to the reference system we can write

$$P_d = \tfrac{1}{2}\lambda v_0^*(t, x) v_0(t, x), \tag{6}$$

where the asterisk denotes complex conjugate. The system can then be considered as a coupling between two subsystems, one described by E and the other described by v. Assuming linear relations between E and v, we have for a coupled system (Askne 1972)

$$\begin{aligned} D_{aa}(\omega, k)E(\omega, k) + D_{ab}(\omega, k)v(\omega, k) &= \mathrm{j}J^s(\omega, k), \\ D_{ba}(\omega, k)E(\omega, k) + D_{bb}(\omega, k, \lambda)v(\omega, k) &= -\mathrm{j}F(\omega, k), \end{aligned} \tag{7}$$

where we have introduced a fictitious force F in order to define the signs of D_{ab} and D_{ba} by the fact that the power delivered to the system is Re v ReF.

Elimination of v yields ($F = 0$)

$$\begin{aligned} D(\omega, k, \lambda)E(\omega, k) &= \left(D_{aa} - \frac{D_{ab}D_{ba}}{D_{bb}}\right)E(\omega, k), \\ A = v(\omega, k)/E(\omega, k) &= -(D_{ba}/D_{bb}). \end{aligned} \tag{8}$$

As the dissipation is described by a damping force λv, λ is included in D_{bb} only as $-\mathrm{j}\lambda$. From (8) we obtain (Askne and Lind 1970)

$$\frac{\partial D}{\partial \lambda} = \frac{D_{ab}D_{ba}}{(D_{bb})^2}\frac{\partial D_{bb}}{\partial \lambda} = -\mathrm{j}A^2\frac{D_{ab}}{D_{ba}}. \tag{9}$$

Since the system is reciprocal we have $D_{ab} = \pm D_{ba}$ (cf Askne 1972) yielding

$$A^2 = \pm \mathrm{j}(\partial D/\partial \lambda). \tag{10}$$

The sign of A^2 is determined from an analysis for large ω and k, when we know that A^2 takes the form $(\mathrm{j}\omega)^{2n}(-\mathrm{j}k)^{2m}B^2$, where B is real. As long as we need A^*A the sign is unimportant. For more illustration of the assumed model of the system see appendix 2 and Askne and Lind (1970).

Assuming the waves with slowly varying amplitudes, equation (8*b*) can be written as a Taylor expansion around ω_0 and $k_{0\mathrm{re}}$.

$$v_0(t, x) = \sum_{n=0}^{+\infty} \frac{A(\omega, k, \lambda)}{n!\,\mathrm{j}^n} \left(\frac{\partial}{\partial \omega}\frac{\partial}{\partial t} - \frac{\partial}{\partial k}\frac{\partial}{\partial x}\right)^n_{\omega_0, k_{0\mathrm{re}}} E_0(t, x). \tag{11}$$

Using now (6) together with (11) we obtain the expression for the dissipated power as

$$P_{\mathrm{d}}(t, x) = \sum_{n=0}^{+\infty} P_{\mathrm{d}}^{(n)}(t, x),$$

$$P_{\mathrm{d}}^{(n)} = \frac{1}{2}\lambda \mathrm{j}^n \sum_{m=0}^{n} \frac{(-1)^m}{m!(n-m)!}\left[A(\omega, k, \lambda)\left(\frac{\overleftarrow{\partial}}{\partial \omega}\frac{\overrightarrow{\partial}}{\partial t} - \frac{\overleftarrow{\partial}}{\partial k}\frac{\overrightarrow{\partial}}{\partial x}\right)^{n-m}_{\omega_0, k_{0\mathrm{re}}} E_0(t, x)\right]^* \tag{12}$$

$$\times\left[A(\omega, k, \lambda)\left(\frac{\partial}{\partial \omega}\frac{\partial}{\partial t} - \frac{\partial}{\partial k}\frac{\partial}{\partial x}\right)^m_{\omega_0, k_{0\mathrm{re}}} E_0(t, x)\right].$$

In the case of quasi-monochromatic waves we obtain the results given in Askne and Lind (1970).

2.2. *The power delivered*

The power delivered by the external source is defined by

$$P_{\mathrm{s}}(t, x) = -\frac{1}{2}\mathrm{Re}(J_0^{\mathrm{s}}(t, x)E_0^*(t, x)). \tag{13}$$

If we multiply (4) and its complex transpose with $E_0^*(t, x)$ and $E_0(t, x)$ respectively and subtract the results, we obtain according to (13)

$$P_{\mathrm{s}}(t, x) = \sum_{n=0}^{+\infty} P_{\mathrm{s}}^{(n)}(t, x),$$

$$P_{\mathrm{s}}^{(n)} = \frac{1}{4n!\,\mathrm{j}^{n-1}}\left\{E_0^*(t, x)\left[D(\omega, k, \lambda)\left(\frac{\overleftarrow{\partial}}{\partial \omega}\frac{\overrightarrow{\partial}}{\partial t} - \frac{\overleftarrow{\partial}}{\partial k}\frac{\overrightarrow{\partial}}{\partial x}\right)^n_{\omega_0, k_{0\mathrm{re}}} E_0(t, x)\right]\right. \tag{14}$$

$$\left. + (-1)^{n+1}E_0(t, x)\left[D(\omega, k, \lambda)\left(\frac{\overleftarrow{\partial}}{\partial \omega}\frac{\overrightarrow{\partial}}{\partial t} - \frac{\overleftarrow{\partial}}{\partial k}\frac{\overrightarrow{\partial}}{\partial x}\right)^n_{\omega_0, k_{0\mathrm{re}}} E_0(t, x)\right]^*\right\}.$$

2.3. *The stored energy and energy flow*

We assume

$$W(t, x) = \sum_{n=0}^{+\infty} W^{(n)}(t, x); \qquad S(t, x) = \sum_{n=0}^{+\infty} S^{(n)}(t, x); \tag{15}$$

and rewrite the energy-conservation relation as

$$\frac{\partial W^{(n-1)}}{\partial t}+\frac{\partial S^{(n-1)}}{\partial x}=P_{\mathrm{s}}^{(n)}-P_{\mathrm{d}}^{(n)}, \qquad n=1,2\ldots \tag{16}$$

where $P_{\mathrm{d}}^{(n)}$ and $P_{\mathrm{s}}^{(n)}$ are given by (12) and (14) respectively. We note from (12) and (14) that the right-hand side of (16) is symmetrical with respect to the derivatives $\partial/\partial\omega$, $\partial/\partial t$ and $\partial/\partial k$, $\partial/\partial x$ and is invariant for changing ω, k, t and x to $-k$, $-\omega$, x and t (or ω, k, t and x to k, ω, $-x$, $-t$) respectively. Consequently, the left-hand side must also be invariant under the same transformations. From the fact that $W^{(n-1)}$ and $S^{(n-1)}$ include the derivatives $\partial^n/\partial\omega^{n-m}\partial k^m$ and $\partial^{n-1}/\partial t^{n-m-1}\partial x^m (m=1,2\ldots n)$, the following symmetry conditions can be written:

(i) $W^{(n-1)}$ can be obtained from $S^{(n-1)}$ and inversely by changing ω, k, t and x to $-k$, $-\omega$, x and t respectively.

(ii) $W^{(n-1)}$ can be obtained from $(-1)^n S^{(n-1)}$ and inversely by changing ω, k, t and x to k, ω, x and t respectively.

It is obvious that both conditions are equivalent. This consideration is connected with the 'temporal' and 'spatial' pictures of pulse propagation in dispersive and absorptive media (cf Anderson *et al* 1975).

Appendix 1 shows how the symmetry conditions together with (8) and (12)–(16) leads to the first-order expressions for the stored energy and the energy flow. The mean stored energy is found to be

$$W(t,x)=W^{(0)}(t,x)+W^{(1)}(t,x)+\ldots$$

with

$$W^{(0)}=\frac{1}{4}\mathrm{Re}\left(\frac{\partial D}{\partial\omega}+2\mathrm{j}\lambda A^*\frac{\partial A}{\partial\omega}\right)E_0^*E_0,$$

$$W^{(1)}=-\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial\omega^2}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega^2}\right)E_0^*\frac{\partial E_0}{\partial t}+\frac{1}{8}\mathrm{j}\left[\frac{\partial^2 D}{\partial\omega\,\partial k}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega\,\partial k}+2\lambda\,\mathrm{Im}\left(\frac{\partial A^*}{\partial\omega}\frac{\partial A}{\partial k}\right)\right]E_0^*\frac{\partial E_0}{\partial x}+\mathrm{cc} \tag{17}$$

(where cc stands for complex conjugate) and the mean energy flow becomes

$$S(t,x)=S^{(0)}(t,x)+S^{(1)}(t,x)+\ldots$$

with

$$S^{(0)}=-\frac{1}{4}\mathrm{Re}\left(\frac{\partial D}{\partial k}+2\mathrm{j}\lambda A^*\frac{\partial A}{\partial k}\right)E_0^*E_0,$$

$$S^{(1)}=-\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial k^2}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial k^2}\right)E_0^*\frac{\partial E_0}{\partial x}+\frac{1}{8}\mathrm{j}\left[\frac{\partial^2 D}{\partial\omega\partial k}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega\,\partial k}+2\lambda\,\mathrm{Im}\left(\frac{\partial A}{\partial\omega}\frac{\partial A^*}{\partial k}\right)\right]E_0^*\frac{\partial E_0}{\partial t}+\mathrm{cc}. \tag{18}$$

The above energy expressions are general for an absorptive and strongly dispersive medium described by a dispersion function $D=D(\omega,k,\lambda)$, and can be extended to second or higher orders. The higher-order derivatives of the envelope function of the wave as well as of the characteristic dispersion function should be important for strong dispersion and/or broad-band signals. If we assume quasi-monochromatic waves and consider the case without spatial dispersion, i.e., D_{ab}, D_{ba} and D_{bb} are wavelength independent, we can obtain from (17)–(18) the results in Askne and Lind (1970).

In the case of a non-absorbing medium we can put $\lambda = 0$ into our energy expressions and get the results in Anderson and Askne (1972).

3. Transport velocity of energy density in an absorbing medium

The energy velocity of monochromatic waves may be defined for linear systems as the ratio of the averages of energy flow to stored energy (cf Brillouin 1960, Loudon 1970). However, for wave pulses the definition of energy velocity must be related to the choice of the initial pulse form. We will define the energy velocity as a quantity representing the rate of change of the 'centre of pulse energy'. If the initial pulse form is $E_0(t, 0)$ and we write the dispersion relation $k = k(\omega)$, it is correct to calculate the 'temporal' energy velocity, i.e. the rate of change of the 'temporal centre of gravity' of the energy flow. However, if the initial pulse form is $E_0(0, x)$, and the dispersion relation is written as $\omega = \omega(k)$, it is natural to work with the 'spatial' energy velocity, i.e. the rate of change of the 'spatial centre of gravity' of the stored energy (cf Anderson and Askne 1974). This is connected with definitions of the 'temporal' or 'spatial' velocity of the pulse maximum (cf Anderson *et al* 1975). However, we will consider here only the 'temporal' energy velocity defined by

$$v_E(x) = \left[\frac{\partial}{\partial x}\left(\frac{\int_{-\infty}^{+\infty} tS(t, x)\,\mathrm{d}t}{\int_{-\infty}^{+\infty} S(t, x)\,\mathrm{d}t}\right)\right]^{-1}. \tag{19}$$

We see that this velocity describes the motion of the 'temporal centre of gravity' of the energy flow and it tells at least part of the story of the flow of the energy. Thus, we have a velocity analogous to the centre of mass velocity of dynamics.

In accordance with the analysis given in § 2 the expressions for the averaged stored energy W and the averaged energy flow S are derived in such a way that the quantity $(P_s - P_d)$ is considered as a driving power for the system. Assuming that the energy velocity is independent of this driving power, it follows from (15) and (16) that equation (19) reduces to

$$v_E(x) = \frac{\int_{-\infty}^{+\infty} S(t, x)\,\mathrm{d}t}{\int_{-\infty}^{+\infty} W(t, x)\,\mathrm{d}t}. \tag{20}$$

The dispersion function describing the wave system is written as

$$D(\omega, k, \lambda) = -d(\omega, k, \lambda)(k - k_{re}(\omega, \lambda) - \mathrm{j}k_{im}(\omega, \lambda)), \tag{21}$$

where we assume that the wave packet is not affected by the other modes, i.e. the variation of the factor $d(\omega, k, \lambda)$ may be neglected. We assume also that $k_{im}(\omega_0, \lambda)$ is one order smaller than $k_{re}(\omega_0, \lambda)$ and that Re d is one order of magnitude larger than Im d. We now introduce the ordering parameter ϵ and write

$$\begin{aligned} (W; S) &= \sum_{n=0}^{+\infty} \epsilon^n (W^{(n)}; S^{(n)}) \\ E_0 &= \sum_{n=0}^{+\infty} \epsilon^n e_n. \end{aligned} \tag{22}$$

The evolution of slowly varying wave packets in absorptive and strongly dispersive media can be studied by a recursive method (cf Anderson *et al* 1975), i.e. a system of coupled first-order differential equations for successive approximations is derived and solved recursively. The amplitudes e_n $(n = 0, 1 \ldots)$ in (21) are found to be (Anderson *et al* 1975)

$$e_0(t, x) = \exp(k_{\mathrm{im}}(\omega_0)x)f(\xi),$$

$$e_1(t, x) = \exp(k_{\mathrm{im}}(\omega_0)x)\mathrm{j}x\left[\frac{1}{2}\left(\frac{\partial^2 k_{\mathrm{re}}}{\partial\omega^2}\right)_{\omega_0} f''(\xi) - \left(\frac{\partial k_{\mathrm{im}}}{\partial\omega}\right)_{\omega_0} f'(\xi)\right],$$

$$e_2(t, x) = \exp(k_{\mathrm{im}}(\omega_0)x)\left\{\frac{1}{6}x\left[\left(\frac{\partial^3 k_{\mathrm{re}}}{\partial\omega^3}\right)_{\omega_0} f'''(\xi) - 3\left(\frac{\partial^2 k_{\mathrm{im}}}{\partial\omega^2}\right)_{\omega_0} f''(\xi)\right]\right. \tag{23}$$

$$\left. -\frac{1}{8}x^2\left[\left(\frac{\partial^2 k_{\mathrm{re}}}{\partial\omega^2}\right)^2_{\omega_0} f^{(4)}(\xi) - 3\left(\frac{\partial^2 k_{\mathrm{re}}}{\partial\omega^2}\right)_{\omega_0}\left(\frac{\partial k_{\mathrm{im}}}{\partial\omega}\right)_{\omega_0} f'''(\xi) + 4\left(\frac{\partial k_{\mathrm{im}}}{\partial\omega}\right)_{\omega_0} f''(\xi)\right]\right\}$$

$$\vdots$$

where f is an arbitrary function which satisfies the initial condition $E_0(t, 0) = f(t)$ and $\xi = t - x(\partial k_{\mathrm{re}}/\partial\omega)_{\omega_0}$. If we choose the input pulse with a Gaussian envelope

$$E_0(t, 0) = f(t) = A\exp(-bt^2), \tag{24}$$

and extend the energy expressions (17) and (18) to second order, we obtain from (20)–(23) the second-order expression for the 'temporal' energy velocity:

$$\frac{1}{v_{\mathrm{E}}} \approx \left(\frac{\partial k_{\mathrm{re}}}{\partial\omega}\right)_{\omega_0} + \lambda C + 2bx\left(\frac{\partial k_{\mathrm{im}}}{\partial\omega}\right)_{\omega_0}\frac{\partial}{\partial\omega}\left(\frac{\partial k_{\mathrm{re}}}{\partial\omega} + \lambda C\right)_{\omega_0} + \frac{1}{2}b\left(\frac{\partial^3 k_{\mathrm{re}}}{\partial\omega^3}\right)_{\omega_0} + \ldots \tag{25a}$$

with

$$C = \frac{[(\partial k_{\mathrm{im}}/\partial\lambda)(\partial^2 k_{\mathrm{re}}/\partial\omega\partial\lambda)] - [(\partial k_{\mathrm{re}}/\partial\lambda)(\partial^2 k_{\mathrm{im}}/\partial\omega\partial\lambda)]}{[(\partial k_{\mathrm{re}}/\partial\lambda)^2 + (\partial k_{\mathrm{im}}/\partial\lambda)^2]^{1/2}}. \tag{25b}$$

According to the recursive method, expression (25*a*) is valid for limited distances

$$x < x_{\mathrm{crit}} \sim \left[b\left|\left(\frac{\partial^2 k_{\mathrm{re}}}{\partial\omega^2}\right)_{\omega_0}\right| + b^{1/2}\left|\left(\frac{\partial k_{\mathrm{im}}}{\partial\omega}\right)_{\omega_0}\right|\right]^{-1}. \tag{26}$$

It is interesting to note that the energy velocity given by (25) depends not only on the parameters of the medium but also on the length of the propagation path, as well as on the width of the pulse. It appears from (25*a*) that the x dependence is connected with the second-order dispersive and absorptive effects. As the distance x increases, the change in the fine-structure frequency (i.e. the spectrum of the pulse) causes a variation in the velocity v_{E}. In fact, each spectral interval $(\omega, \omega + \mathrm{d}\omega)$ is associated with its own energy velocity which differs in each interval $(x, x + \mathrm{d}x)$. Thus, we can define the 'temporal' energy velocity averaged over a distance from the origin $x = 0$ up to the point x, which is equal to (25). We note also that equation (25) includes the following relations.

(a) By assuming monochromatic waves, $b = 0$, equation (25) is reduced to

$$\frac{1}{v_{\mathrm{E}}} \approx \left(\frac{\partial k_{\mathrm{re}}}{\partial\omega}\right)_{\omega_0} + \lambda C, \tag{27}$$

i.e. the energy velocity is equal to the classical group velocity complemented by the correction term proportional to the dissipational parameter λ. A property of the energy velocity which is required by the theory of relativity is that it should be smaller than the free-space velocity of light c at all frequencies ω_0. It is not immediately obvious from (26) that v_E has this property. The property can be verified, however, for a given dispersion relation $k = k(\omega, \lambda)$ describing an absorbing medium (see § 3). It can be shown that in the case of wave propagation through an absorbing dielectric for frequencies near the resonance frequency, the expression (27) leads to the result obtained by Loudon (1970).

(*b*) The second-order expression for the 'temporal' velocity of propagation of a Gaussian pulse through an absorbing medium is given by (cf Anderson *et al* 1975)

$$\frac{1}{v_t^{(2)}} \approx \left(\frac{\partial k_{re}}{\partial \omega}\right)_{\omega_0} + 2bx\left(\frac{\partial k_{im}}{\partial \omega}\right)_{\omega_0}\left(\frac{\partial^2 k_{re}}{\partial \omega^2}\right)_{\omega_0}, \tag{28}$$

i.e. the velocity of the temporal maximum of the envelope function at a given distance. If we neglect in (25) the correction terms proportional to λ and assume $(\partial^2 k_{re}/\partial\omega^2)_{\omega_0} \to 0$, we obtain the expression (28). This shows that the energy velocity in an absorbing medium always differs from the velocity of the pulse maximum. The difference depends on the correction term λC.

(*c*) In the case of a non-absorbing medium, $\lambda = 0$ and $k_{im}(\omega_0) = 0$, the relation (25) yields the second-order velocity of inertia introduced in Anderson and Askne (1974).

4. Pulse velocity and energy velocity in an atomic medium

Let us now consider an atomic medium described by the dispersion relation

$$k(\omega, \lambda) \approx \frac{\omega}{c}\left(1 - \frac{\omega_p^2}{2\omega(\omega - \Omega^2/\omega - j\lambda/Nm)}\right), \qquad \omega_p \ll |Nm/\lambda|, \tag{29}$$

where $\omega_p^2 = Ne^2/m\varepsilon_0$, N is the density of electrons, Ω is the resonance frequency, $\lambda = Nm/T_2$ is the loss factor and T_2 is the relaxation time. We assume a non-inverted medium ($\omega_p^2 > 0$) and that the pulse centre frequency ω_0 is equal to the atomic line centre frequency ω_0 which is also equal to the atomic line centre frequency Ω.

4.1. *The 'temporal' pulse velocity*

From (28) we can calculate the expression for the 'temporal' pulse velocity, which is in this case equal to the group velocity (cf Anderson *et al* 1975):

$$v_t^{(2)} = v_g = \frac{c}{1-(\omega_p T_2)^2}, \qquad \omega_p^2 > 0, \tag{30}$$

which can exceed the free-space velocity of light c. However, it should be noted that when the pulse velocity exceeds c (i.e. when the velocity of the maximum exceeds the velocity of the pulse front), steepening effects eventually invalidate the assumption of slowly varying amplitudes. A qualitative condition is that the propagation path should be less than the pulse width divided by $(1/c - 1/v_t^{(2)})$ (see Anderson *et al* 1975). A physical interpretation of the fact that v_t may be greater than c is of basic interest in connection with experiments. Crisp (1971) suggested that the asymmetric absorption of energy of the pulse results in a forward motion of the centre of gravity of the pulse so that the pulse maximum is moving with speed greater than c. However, no signal or energy will propagate faster than the free-space velocity of light c. We will examine this idea for an atomic medium starting from the expression for the power dissipated, P_d. Assuming a medium without spatial dispersion we have from (12)

$$P_d = P_d^{(0)} + P_d^{(1)} + \ldots = \frac{1}{2}\lambda\left\{\left|\left(j\frac{\partial D}{\partial \lambda}\right)^{1/2}\right|^2 E_0^2 + 2\mathrm{Im}\left[\left(j\frac{\partial D}{\partial \lambda}\right)^{1/2*}\frac{\partial}{\partial \omega}\left(j\frac{\partial D}{\partial \lambda}\right)^{1/2}\right]E_0\frac{\partial E_0}{\partial t}\right\} \quad (31)$$

where

$$E_0 = e_0 = \exp(k_{\mathrm{im}}(\omega_0)x)f(t - x/v_g) \qquad \text{and} \qquad v_g = (\partial k_{\mathrm{re}}/\partial \omega)_{\omega_0}^{-1}$$

see (22). The dispersion function for an atomic medium can be written as (Askne and Lind 1970):

$$D(\omega, k, \lambda) = \varepsilon_0\omega\left(1 - \frac{\omega_p^2}{\omega(\omega - \Omega^2/\omega - j\lambda/Nm)}\right) - \frac{k^2}{\omega\mu_0}, \qquad \omega_p^2 > 0, \quad (32)$$

and the power dissipated becomes

$$P_d = \frac{1}{2}\varepsilon_0\omega_p^2 T_2\left(E_0^2 - 4T_2E_0\frac{\partial E_0}{\partial t}\right), \quad (33)$$

where we have assumed $\omega_0 = \Omega$.

The temporal maximum of the envelope function $|E_0(t, x)|$ at a given distance x is determined by $\partial|E_0(t, x)|/\partial t|_{t_M} = 0$, where $t_M = x/v_g$ and we obtain

$$P_{dM} = \tfrac{1}{2}\varepsilon_0\omega_p^2 T_2 E_0^2. \quad (34)$$

The power P_d associated with the leading half of the pulse ($\partial E_0/\partial t > 0$) is according to (33) less than the power dissipated in the trailing half of the pulse ($\partial E_0/\partial t < 0$). Thus, more energy is absorbed from the trailing half of the pulse than from the leading half and this results in a motion of the pulse maximum at a velocity greater than c.

4.2. *The energy velocity*

Using (25) together with (29) and assuming that $(\partial^3 k_{\mathrm{re}}/\partial\omega^3)_{\omega_0}$ is negligible, the energy velocity obtained is

$$v_E \approx \frac{c}{1 + (\omega_p T_2)^2}, \quad (35)$$

which is always less than c for $\omega_p^2 > 0$. Thus, it seems that the terms in (25) proportional to the loss factor λ compensate the pulse velocity (27) so that the resulting energy velocity is always less than c for a non-inverted medium. If we assume monochromatic waves and introduce the complex dielectric constant

$$\varepsilon^{1/2} = kc/\omega = n - j\varkappa, \quad (36)$$

where n is the index of refraction and $\varkappa$ is the extinction coefficient, we find from (26) and (29) that the expression for energy velocity can be written as

$$v_E = \frac{c}{n + 2\omega T_2\varkappa}, \quad (37)$$

which is the result obtained in Loudon (1970). However, it should be noted that for an inverted medium ($\omega_p^2 < 0$) the resulting energy velocity can be greater than the velocity of light or negative as a consequence of the relation (20), where the mean stored energy W can be negative.

5. Conclusions

The results obtained above expose the laws of propagation of pulses with slowly varying amplitudes in the presence of absorption and dispersion. The essential feature of our analysis is to demonstrate how energy expressions can be derived from the dispersion function if this is an explicit function of the dissipational parameter. The resulting first-order expressions for the stored energy, energy flow and dissipated power can be extended to include higher-order dispersive and absorptive correction terms. The expression for the 'temporal' energy velocity is obtained by using the results from the recursive method and is valid only for a propagation path limited by a critical value. The results of the present work are restricted to the case of moderate absorption but can be applied to wave pulses in media with temporal as well as spatial dispersion. The present analysis may also be of value in connection with a discussion of the signal velocity.

Acknowledgments

The author gratefully acknowledges stimulating discussions with Drs J Askne and D Anderson.

Appendix 1. Derivation of stored energy and energy flow

A.1. Derivation of $W^{(0)}$ and $S^{(0)}$

From the relation (16) for $n = 1$ we have

$$\frac{\partial W^{(0)}}{\partial t} + \frac{\partial S^{(0)}}{\partial x} = P_s^{(1)} - P_d^{(1)} \tag{A.1}$$

where $P_d^{(1)}$ and $P_s^{(1)}$ are obtained from (12) and (14) respectively as

$$\begin{aligned} P_d^{(1)} &= \frac{1}{2} \mathrm{j}\lambda\left[\left(A\frac{\partial A^*}{\partial \omega} E_0 \frac{\partial E_0^*}{\partial t} - A^* \frac{\partial A}{\partial \omega} E_0^* \frac{\partial E_0}{\partial t}\right) - \left(A\frac{\partial A^*}{\partial k} E_0 \frac{\partial E_0^*}{\partial x} - A^* \frac{\partial A}{\partial k} E_0^* \frac{\partial E_0}{\partial x}\right)\right] \\ P_s^{(1)} &= \frac{1}{4}\left[\left(\frac{\partial D}{\partial \omega} E_0^* \frac{\partial E_0}{\partial t} + \frac{\partial D^*}{\partial \omega} E_0 \frac{\partial E_0^*}{\partial t}\right) - \left(\frac{\partial D}{\partial k} E_0^* \frac{\partial E_0}{\partial x} + \frac{\partial D^*}{\partial k} E_0 \frac{\partial E_0^*}{\partial x}\right)\right] \end{aligned} \tag{A.2}$$

with $A(\omega, k, \lambda) = (\mathrm{j}(\partial D/\partial \lambda))^{1/2}$. Using now (8) together with (A.2), we obtain

$$\begin{aligned} P_s^{(1)} - P_d^{(1)} &= \frac{1}{4}\left(\frac{\partial D_{aa}}{\partial \omega} - 2\frac{|D_{ab}| \mathrm{Re}\, D_{bb}}{|D_{bb}|^2} \frac{\partial |D_{ab}|}{\partial \omega} + \frac{|D_{ab}|^2}{|D_{bb}|^2} \frac{\partial D_{bb}}{\partial \omega}\right) \frac{\partial}{\partial t}(E_0^* E_0) \\ &\quad - \frac{1}{4}\left(\frac{\partial D_{aa}}{\partial k} - 2\frac{|D_{ab}| \mathrm{Re}\, D_{bb}}{|D_{bb}|^2} \frac{\partial |D_{ab}|}{\partial k} + \frac{|D_{ab}|^2}{|D_{bb}|^2} \frac{\partial D_{bb}}{\partial k}\right) \frac{\partial}{\partial x}(E_0^* E_0), \end{aligned} \tag{A.3}$$

which can be written as

$$P_s^{(1)}-P_d^{(1)}=\frac{1}{4}\mathrm{Re}\left(\frac{\partial D}{\partial \omega}+2\mathrm{j}\lambda A^*\frac{\partial A}{\partial \omega}\right)\frac{\partial}{\partial t}(E_0^*E_0)-\frac{1}{4}\mathrm{Re}\left(\frac{\partial D}{\partial k}+2\mathrm{j}\lambda A^*\frac{\partial A}{\partial k}\right)\frac{\partial}{\partial x}(E_0^*E_0). \quad \text{(A.4)}$$

Thus, it follows from (A.1) that the zero-order expressions for the mean stored energy and the mean energy flow are

$$\begin{aligned} W^{(0)}&=\frac{1}{4}\mathrm{Re}\left(\frac{\partial D}{\partial \omega}+2\mathrm{j}\lambda A^*\frac{\partial A}{\partial \omega}\right)E_0^*E_0,\\ S^{(0)}&=-\frac{1}{4}\mathrm{Re}\left(\frac{\partial D}{\partial k}+2\mathrm{j}\lambda A^*\frac{\partial A}{\partial k}\right)E_0^*E_0, \end{aligned} \quad \text{(A.5)}$$

where $A=[\mathrm{j}(\partial D/\partial\lambda)]^{1/2}$.

A.2. Derivation of $W^{(1)}$ and $S^{(1)}$

Substituting $n=2$ into (16) yields

$$\frac{\partial W^{(1)}}{\partial t}+\frac{\partial S^{(1)}}{\partial x}=P_s^{(2)}-P_d^{(2)}, \quad \text{(A.6)}$$

where $P_d^{(2)}$ and $P_s^{(2)}$ are given by (10) and (13) respectively as

$$\begin{aligned} P_d^{(2)}=\frac{1}{4}\lambda\bigg(&-A^*\frac{\partial^2 A}{\partial\omega^2}E_0^*\frac{\partial^2 E_0}{\partial t^2}+2A^*\frac{\partial^2 A}{\partial\omega\partial k}E_0^*\frac{\partial^2 E_0}{\partial t\partial x}-A^*\frac{\partial^2 A}{\partial k^2}E_0^*\frac{\partial^2 E_0}{\partial x^2}+\frac{\partial A^*}{\partial\omega}\frac{\partial A}{\partial\omega}\frac{\partial E_0}{\partial t}\frac{\partial E_0}{\partial t}\\ &-2\frac{\partial A^*}{\partial\omega}\frac{\partial A}{\partial k}\frac{\partial E_0^*}{\partial t}\frac{\partial E_0}{\partial x}+\frac{\partial A^*}{\partial k}\frac{\partial A}{\partial k}\frac{\partial E_0^*}{\partial x}\frac{\partial E_0}{\partial x}\bigg)+\mathrm{cc} \end{aligned} \quad \text{(A.7)}$$

$$P_s^{(2)}=-\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial\omega^2}\frac{\partial^2 E_0}{\partial t^2}-2\frac{\partial^2 D}{\partial\omega\partial k}\frac{\partial^2 E_0}{\partial t\partial x}+\frac{\partial^2 D}{\partial k^2}\frac{\partial^2 E_0}{\partial x^2}\right)E_0^*+\mathrm{cc}.$$

We assume that the expressions for $W^{(1)}$ and $S^{(1)}$ have such forms that $W^{(1)}$ can be obtained from $S^{(1)}$ and inversely by changing ω, k, t and x to k, ω, x and t respectively. We rewrite (A.7) and obtain

$$\begin{aligned} P_s^{(2)}-P_d^{(2)}=&-\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial\omega^2}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega^2}\right)\frac{\partial}{\partial t}\left(E_0^*\frac{\partial E_0}{\partial t}\right)-\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial k^2}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega^2}\right)\frac{\partial}{\partial x}\left(E_0^*\frac{\partial E_0}{\partial x}\right)\\ &+\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial\omega\partial k}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega\partial k}+\Gamma_1\right)\frac{\partial}{\partial t}\left(E_0^*\frac{\partial E_0}{\partial x}\right)\\ &+\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial\omega\partial k}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega\partial k}+\Gamma_2\right)\frac{\partial}{\partial x}\left(E_0^*\frac{\partial E_0}{\partial t}\right)\\ &+\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial\omega^2}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega^2}+2\mathrm{j}\lambda\frac{\partial A^*}{\partial\omega}\frac{\partial A}{\partial\omega}\right)\frac{\partial E_0^*}{\partial t}\frac{\partial E_0}{\partial t}\\ &+\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial k^2}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial k^2}+2\mathrm{j}\lambda\frac{\partial A^*}{\partial k}\frac{\partial A}{\partial k}\right)\frac{\partial E_0^*}{\partial x}\frac{\partial E_0}{\partial x} \end{aligned}$$

$$-\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial\omega\partial k}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega\partial k}+\Gamma_1+2\mathrm{j}\lambda\frac{\partial A^*}{\partial\omega}\frac{\partial A}{\partial k}\right)\frac{\partial E_0^*}{\partial t}\frac{\partial E_0}{\partial x}$$
$$-\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial\omega\partial k}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega\partial k}+\Gamma_2+2\mathrm{j}\lambda\frac{\partial A}{\partial\omega}\frac{\partial A^*}{\partial k}\right)\frac{\partial E_0}{\partial t}\frac{\partial E_0^*}{\partial x}$$
$$-\frac{1}{8}\mathrm{j}(\Gamma_1+\Gamma_2)E_0^*\frac{\partial^2 E_0}{\partial t\partial x}+\mathrm{cc}. \tag{A.8}$$

From (8) it can be shown that

$$\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial\omega^2}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega^2}+2\mathrm{j}\lambda\frac{\partial A^*}{\partial\omega}\frac{\partial A}{\partial\omega}\right)+\mathrm{cc}=0 \tag{A.9}$$

and similarly

$$\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial k^2}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial k^2}+2\mathrm{j}\lambda\frac{\partial A^*}{\partial k}\frac{\partial A}{\partial k}\right)+\mathrm{cc}=0. \tag{A.10}$$

We also find that

$$2\lambda\,\mathrm{Re}\left(\frac{\partial A^*}{\partial\omega}\frac{\partial A}{\partial k}\right)=-\mathrm{Im}\left(\frac{\partial^2 D}{\partial\omega\partial k}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega\partial k}\right),$$
$$\mathrm{Im}\left(\frac{\partial A^*}{\partial\omega}\frac{\partial A}{\partial k}\right)=-\mathrm{Im}\left(\frac{\partial A}{\partial\omega}\frac{\partial A^*}{\partial k}\right). \tag{A.11}$$

Thus

$$\frac{1}{4}\left[\mathrm{Im}\left(\frac{\partial^2 D}{\partial\omega\partial k}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega\partial k}\right)+2\lambda\frac{\partial A^*}{\partial\omega}\frac{\partial A}{\partial k}-\mathrm{j}(\Gamma_1-\Gamma_2^*)\right]=0 \tag{A.12}$$

if

$$\Gamma_1=-\Gamma_2=2\lambda\ \mathrm{Im}\left(\frac{\partial A^*}{\partial\omega}\frac{\partial A}{\partial k}\right). \tag{A.13}$$

Using (A.8)–(A.13) together with (A.6), we find

$$W^{(1)}=-\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial\omega^2}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega^2}\right)E_0^*\frac{\partial E_0}{\partial t}$$
$$+\frac{1}{8}\mathrm{j}\left[\frac{\partial^2 D}{\partial\omega\partial k}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega\partial k}+2\lambda\,\mathrm{Im}\left(\frac{\partial A^*}{\partial\omega}\frac{\partial A}{\partial k}\right)\right]E_0^*\frac{\partial E_0}{\partial x}+\mathrm{cc},$$
$$S^{(1)}=-\frac{1}{8}\mathrm{j}\left(\frac{\partial^2 D}{\partial k^2}+2\mathrm{j}\lambda A^*\frac{\partial A}{\partial k^2}\right)E_0^*\frac{\partial E_0}{\partial x}$$
$$+\frac{1}{8}\mathrm{j}\left[\frac{\partial^2 D}{\partial\omega\partial k}+2\mathrm{j}\lambda A^*\frac{\partial^2 A}{\partial\omega\partial k}+2\lambda\,\mathrm{Im}\left(\frac{\partial A}{\partial\omega}\frac{\partial A^*}{\partial k}\right)\right]E_0^*\frac{\partial E_0}{\partial t}+\mathrm{cc}. \tag{A.14}$$

Finally, we note that this procedure can be extended to give higher-order energy expressions.

Appendix 2. Illustration of assumed model of the medium

In order to illustrate the assumed model of the medium we apply the analysis given in § 2.1 to the electromagnetic wave propagation in a cold electron plasma with collisions. The system is described by the Maxwell's equations together with the force equation:

$$\nabla\times\boldsymbol{E}=-\mu_0\frac{\partial\boldsymbol{H}}{\partial t},$$

$$\nabla\times\boldsymbol{H}=\varepsilon_0\frac{\partial\boldsymbol{E}}{\partial t}-Ne\boldsymbol{v}+\boldsymbol{J}^{\mathrm{s}},\qquad(\mathrm{A}.15)$$

$$Nm\left(\frac{\partial\boldsymbol{v}}{\partial t}+\nu\boldsymbol{v}\right)=-Ne\boldsymbol{E}+\boldsymbol{F},$$

where $-e$ is the charge, m the mass and N the density of electrons, ν is the collision frequency. $\boldsymbol{J}^{\mathrm{s}}$ and $\boldsymbol{F}$ are the external current density and the fictitious force respectively. Assuming propagation in the x direction and $\boldsymbol{E}=\hat{\boldsymbol{y}}E$, the Fourier transformation of (A.15) yields

$$\left(\omega\varepsilon_0-\frac{k^2}{\omega\mu_0}\right)E-\mathrm{j}Nev=\mathrm{j}J^{\mathrm{s}},$$

$$\mathrm{j}NeE+Nm(\omega-\mathrm{j}\nu)v=-\mathrm{j}F,\qquad(\mathrm{A}.16)$$

which corresponds to equation (7). We identify $\lambda=Nm\nu$ and obtain ($F=0$):

$$DE=\left(\omega\varepsilon_0-\frac{\omega_{\mathrm{p}}^2\varepsilon_0}{\omega-\mathrm{j}\nu}-\frac{k^2}{\omega\mu_0}\right)E=\mathrm{j}J^{\mathrm{s}}$$

$$A^2=-\mathrm{j}\frac{\partial D}{\partial\lambda}=-\frac{\omega_{\mathrm{p}}^2\varepsilon_0}{Nm(\omega-\mathrm{j}\nu)^2},\qquad(\mathrm{A}.17)$$

where $\omega_{\mathrm{p}}^2=Ne^2/m\varepsilon_0$. According to (14), (17) and (18) the zero-order averaged expressions for dissipated power, stored energy and energy flow are respectively

$$P_{\mathrm{d}}^{(0)}=\frac{\varepsilon_0\nu}{2}\frac{\omega_{\mathrm{p}}^2}{\omega^2+\nu^2}E_0^*E_0,$$

$$W^{(0)}=\frac{1}{4}\left(\varepsilon_0+\frac{\omega_{\mathrm{p}}^2\varepsilon_0}{\omega^2+\nu^2}+\frac{k_{\mathrm{re}}^2}{\omega^2\mu_0}\right)E_0^*E_0,\qquad(\mathrm{A}.18)$$

$$S^{(0)}=\frac{1}{2}\frac{k_{\mathrm{re}}}{\omega\mu_0}E_0^*E_0,$$

which would be expected by physical arguments. Finally, we note that the higher-order energy expressions can be calculated easily from (14), (17) and (18) together with (A.17). For more examples in these matters see Askne and Lind (1970).

References

Anderson D and Askne J 1972 *Proc. IEEE (Lett.)* **60** 901–2
——— 1974 *Proc. IEEE* **62** 1518–28
Anderson D, Askne J and Lisak M 1975 *Phys. Rev.* A **12** 1546–52

Askne J 1972 *Int. J. Electron.* **32** 573–91
Askne J and Lind B 1970 *Phys. Rev.* A **2** 2335–40
Brillouin L 1960 *Wave Propagation and Group Velocity* (New York: Academic)
Crisp M D 1971 *Phys Rev.* A **4** 2104–8
Ginzburg V L 1964 *Propagation of Electromagnetic Waves in Plasmas* (Oxford, London: Pergamon)
Loudon R 1970 *J Phys. A: Gen Phys.* **3** 233–45